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Existing lattice data on the QCD phase transition are re-analyzed. $T_c/\Lambda_{\overline{MS}}$ is found to scale for lattices with only 3 time slices. As a result, this ratio can be calculated with a precision of better than four parts in a thousand in the pure gauge theory. Another implication of scaling is that the equation of state can be found reliably on fairly coarse lattices. We also give a preliminary estimate of $T_c/\Lambda_{\overline{MS}}$ for QCD with dynamical quarks.

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The QCD phase transition temperature (T_c) is a fundamental constant of the hadronic world, and will soon be accessible to experiments. It has also been the target of many lattice computations. Current practice is to express T_c in units of the mass of the rho meson (M_ρ) or the square root of the string tension ($\sqrt{\sigma}$) [2]. However, in the last few years it has become clear that α_S measured on fairly coarse lattices [3] yield results comparable to those obtained in precision measurements at LEP and in other experiments [4]. This prompts us to test the approach to the continuum limit of QCD thermodynamics by testing the constancy of $T_c/\Lambda_{\overline{MS}}$, where $\Lambda_{\overline{MS}}$ is the QCD scale parameter extracted in the $\overline{\rm MS}$ scheme.

In the limit when all quark masses are zero (or infinite), QCD has only one dimensionless parameter— the coupling, $\alpha_S = g^2/4\pi$. Quantum corrections transmute it into a momentum scale. This scale is either given explicitly as the QCD parameter Λ, or specified implicitly, as the value of the coupling $\alpha_S(\mu)$ at scale μ.

In the lattice regularisation of QCD, the value of the lattice spacing (a) is fixed in this way by the bare coupling $\beta = 6/g^2$. However, $6/\beta$ is not a good expansion parameter for perturbation theory. It seems more appropriate to define the coupling through some physically motivated process. In one definition [3], α_S at scale $3.4018/a$ is found from the logarithm of the plaquette value ($\mathcal{P}$) [5] through the formula

$$-\ln\mathcal{P} = \frac{4\pi}{3}\alpha_V\left(1-(1.1897+0.071N_f)\alpha_V\right). \tag{1}$$

We could choose to work in either this V-scheme, or in the usual $\overline{\rm MS}$ scheme whose relation to this is known [6]. In the V-scheme at 2-loop order we can write

$$a\Lambda_V = 3.4018R\left(1/4\pi\beta_0\alpha_V\right) \tag{2}$$

where $R^2(x) = \exp(-x)x^{\beta_1/\beta_0^2}$. The β-function, $\overline{\beta}(g)$, can be written as the inverse of the derivative of $\log R$ with respect to g.

The QCD phase transition temperature is determined by tuning the bare coupling on lattices with $N_t \ll N_s$ (where N_t is the number of sites in the Euclidean time direction and N_s that in the spatial directions). Then,

$$T = 1/aN_t, \tag{3}$$

where a is the lattice spacing at the coupling where the phase transition occurs. Our strategy is to compute the renormalised coupling α_V at these bare couplings and hence determine $T/\Lambda_{\overline{MS}}$ using eqs. (2,3), and the known value of $\Lambda_{\overline{MS}}/\Lambda_V$ [3].

In quenched QCD with the usual Wilson action, the critical bare couplings, β_c, have been determined for $2 \le N_t \le 16$ [7]. The main source of systematic uncertainty in the older data arises from the fact that the thermodynamic limit $N_s \to \infty$ was not taken. Later data [8–10] have taken this limit, and we only used these to study scaling. The statistical errors in these later studies are also much smaller, and hence they are able to test the scaling hypothesis much more stringently.

We extracted α_V from plaquette measurements tabulated in the literature [9], supplemented by some new measurements. Values of $\ln\mathcal{P}$ at β_c were obtained by cubic spline interpolation of this combined set. Statistical errors in the interpolated values were found by propagation. We probed systematic errors in the interpolation by the change in α_V or removal of some of the knot points. Both these errors are small compared to the uncertainty in β_c. The results of our analysis are shown in Figure 1. $T_c/\Lambda_{\overline{MS}}$ is constant down to β_c for $N_t = 3$.

More detailed results are shown in the right hand panel of Figure 1. Small scaling violations at these couplings, seen in measurements at $T = 0$, have been attributed to finite lattice spacing errors [11]. On replacing the scaling function R in eq. (2) by

$$\overline{R}(\alpha_V) = R(\alpha_V)\left[1+c_2\hat{a}^2+c_4\hat{a}^4\right] \tag{4}$$

where $\hat{a} = R(\alpha_V)/R(\alpha_V^0)$ and α_V^0 is determined at $\beta = 6$, it was found [12] that data on $a\sqrt{\sigma}$ and the Sommer scale, r_0/a, could be quantitatively described down to $\beta = 5.4$. Using the parametrisation of [12] for the form in eq. (4) we extracted $T_c/\Lambda_{\overline{MS}}$ again. A constant value

$$T_c/\Lambda_{\overline{MS}} = 1.132 \pm 0.004 \tag{5}$$

could be fitted to the data down to $N_t = 3$ with $\chi^2 = 3.8$ for 4 degrees of freedom. The dark band in the figure shows the best fit.

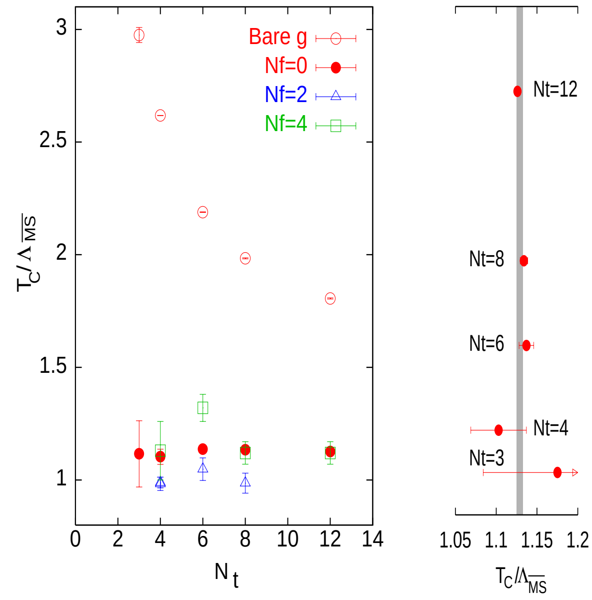

FIG. 1. The panel on the left shows $T_c/\Lambda_{\overline{MS}}$ as a function of N_t. For $N_f = 0$ we show tests of scaling for both the bare lattice coupling and α_V. For $N_f = 2$ with $m/T_c = 0.1$ and $N_f = 4$ with $m/T_c = 0.08$ only the latter is shown. The panel on the right shows details of $T_c/\Lambda_{\overline{MS}}$ using the scale function in eq. (4).

Using this best fit value of $T_c/\Lambda_{\overline{MS}}$ and an extraction of $\Lambda_{\overline{MS}} = 247 \pm 16$ MeV in pure gauge theory [12], we find that

$$T_c = 280 \pm 1 \pm 18 \text{ MeV}. \tag{6}$$

The first error comes from the determination of the ratio $T_c/\Lambda_{\overline{MS}}$ and the second is due to the error in the determination of $\Lambda_{\overline{MS}}$. Although it is possible to add these errors in quadrature, we have displayed them in this manner to emphasise the sources of error. We note that this value of T_c, derived from simulations using a Wilson gauge action, is 1.4% higher than that found in a study using an RG-improved action [13].

As a result of this analysis it is clear that perturbative formulae can be used in the extraction of thermodynamic quantities in the QCD plasma, provided that a good scheme is used to determine the QCD coupling. We examine the energy density (E) and the pressure (P). These can be written in terms of the difference $\Delta_i = \mathcal{P}_i - \mathcal{P}_0$ between the spatial ($i = s$) and temporal ($i = t$) plaquettes and their zero temperature counterpart $\mathcal{P}_0$. E is defined by the formula

$$\frac{E}{T^4} = 6N_cN_t^4\left[\frac{\Delta_s - \Delta_t}{4\pi\alpha_V} - (c_s'\Delta_s + c_t'\Delta_t)\right]. \tag{7}$$

The anisotropy coefficients c_s' and c_t' are known to 1-loop order [14]. Since the Δ_i are obtained from plaquettes, α_V should be evaluated at the usual scale of $3.4018/a$. A measure of deviations from ideal gas behaviour is

$$\Delta = \frac{E - 3P}{T^4} = 12N_cN_t^4(c_s' + c_t')(\Delta_t + \Delta_s), \tag{8}$$

which can be combined with eq. (7) to give the pressure. The sum rule $g^3(c_s' + c_t') = \overline{\beta}(g)$ [14], allows us to evaluate Δ beyond 1-loop order. In fact, part of the finite a corrections can be incorporated into Δ by evaluating $\overline{\beta}(g)$ using eq. (4).

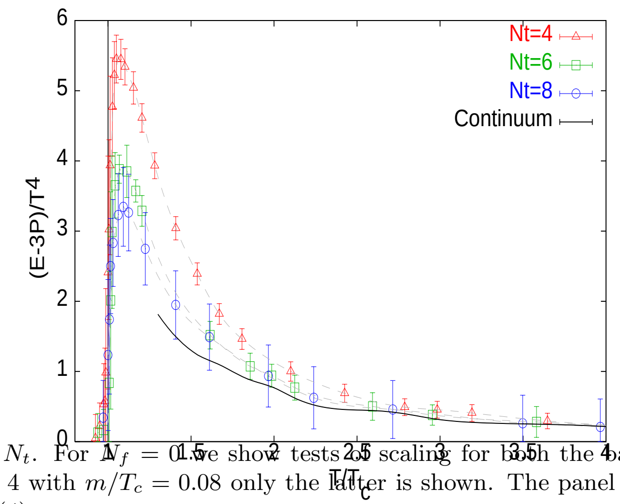

FIG. 2. $(E - 3P)/T^4$ for a QCD plasma evaluated as a function of T
scale. The continuum curve is not continued into the region where finite

Raw data on $\mathcal{P}_{0,s,t}$ at a range of couplings measured on $N_t \times (4N_t)^3$ lattices ($N_t = 4$, 6 and 8) are tabulated in [9]. We have used these to evaluate thermodynamic quantities only for $T < 4T_c$, because the finite spatial volumes of the lattices used may cause spatial deconfinement above $4T_c$. As expected, Δ varies as $1/N_t^2$ at fixed T, *i.e.*, as a^2. It shows a peak at $T \approx 1.1T_c$ in the continuum limit. However, the location of the peak is uncertain because there is no unique definition of T_c at finite volumes. Different definitions, which all coincide in the thermodynamic limit, give different values of the pseudo-critical point on finite volume systems [15]. Using coarse lattices, we have estimated that in $SU(3)$ theory for $N_s = 4N_t$ this inherent uncertainty in the critical coupling may be as much as $\delta\beta \approx 0.005$–$0.01$: much larger than the statistical error for any given definition of the critical point. For similar reasons the value of Δ at the peak cannot be reliably extracted without taking into account finite volume effects.

In the range $1.3T_c \le T \le 4T_c$, finite volume corrections are expected to be small. In this range Δ is monotonically decreasing, implying that at sufficiently high temperatures the QCD plasma is weakly interacting. However, many quantities of interest can be extracted only near T_c. The results of a finite size scaling study which does this will be reported elsewhere.

We have also evaluated E/T^4 and found that it scales to the continuum limit as $1/N_t^2$. In the temperature range where finite volume effects are strong, the large value of Δ gives a negative value to P evaluated using the formulae in eqs. (7,8). As we have argued already, an evaluation of thermodynamics in this range of T requires better control over finite volume effects. However, the problem of negative pressures is also avoided if the continuum limit of P is found by combining E/T^4 and Δ in the limit $a = 0$. This gives nearly vanishing P in the region of the peak in Δ.

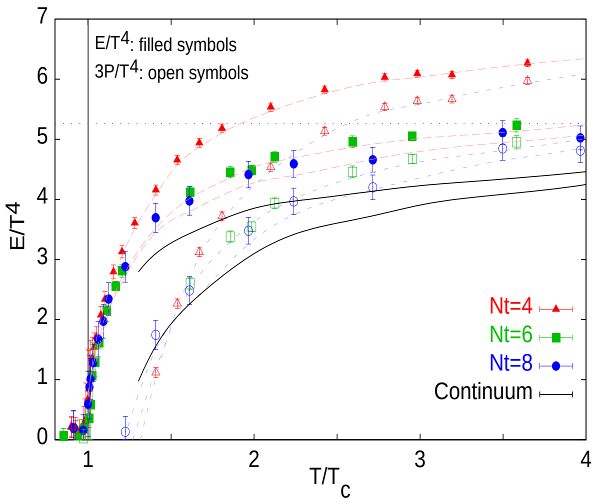

FIG. 3. E/T^4 and $3P/T^4$ for a QCD plasma evaluated as explained in the text, using the data of [9]. The dashed curves are smooth cubic spline fits. The dotted horizontal line is the ideal gas result in the continuum limit.

In Figure 3 we show our estimates of the continuum limit of E and P in the range $1.3T_c \leq T \leq 4T_c$, where finite volume effects are small. At $4T_c$ our estimate of the energy density is about 15% lower (and the pressure about 19% lower) than for an ideal gas. At this temperature, our estimates of E and P are about 5% lower than the earlier determination which used the integral method [9]. This can be attributed entirely to the differing treatment of the region around the peak of Δ. The speed of sound in the QCD plasma (c_s) is slightly lower than the ideal gas value ($c_s^2 = 1/3$) at $4T_c$ (but consistent with it within errors), and falls to $c_s^2 \approx 0.1 \pm 0.025$, at $1.5T_c$.

The observation of scaling on coarse lattices would pay off in the study of finite temperature QCD with dynamical light quarks. Since there are inherent difficulties with simulating QCD with dynamical massless quarks, all lattice studies have used quarks with mass $m > 0$. In such theories, scaling should be tested at fixed $m/\Lambda_{\overline{MS}}$ (or equivalently, fixed ratio of m and any hadronic mass scale).

Measurements of β_c in QCD with 4 flavours of dynamical staggered quarks have been performed for $N_t = 4$, 6 [16] and 8 [17]. This last measurement was done with $m/T_c = 0.08$. The simulations at smaller N_t were done at several values of m, enabling us to find β_c at $m/T_c = 0.08$ by interpolation. Plaquette values were taken from a recent finite temperature simulation at fixed m/T_c [18]. Since it is known that $\Delta_{t,s}/\mathcal{P}_0$ are less than 0.1%, even near T_c, we have used $(\mathcal{P}_s + \mathcal{P}_t)/2$ to determine α_V. The difference of α_V measured using $\mathcal{P}_s$ and $\mathcal{P}_t$ is taken as an estimate of its error. We used eq. (2) to set the scale. $T_c/\Lambda_{\overline{MS}}$, when determined through the bare coupling, changes from 8.2 to 4.7 in going from $N_t = 4$ to 8. As shown in Figure 1, when α_V is used, 2-loop scaling works much better.

The phase transition in 2-flavour QCD has been studied in greater detail (see [2] for a recent compilation of data). Measurements of β_c using dynamical staggered quarks exist for $N_t \leq 12$ [19,20]. Earlier simulations with Wilson quarks (which have order a lattice artifacts) showed that pion masses were rather high compared to those obtained with staggered quarks. This problem becomes less acute on using improved actions for Wilson quarks, and finite temperature simulations have now been performed with such improved actions [22,23]. There have also been some studies with domain wall Fermions.

We have fixed α_V and set the scale using published plaquette values for staggered quarks at several bare quark masses [20,21]. $T_c/\Lambda_{\overline{MS}}$ for $N_f = 2$ shown in Figure 1 are based on the subset of the data which uses staggered quarks at $m/T_c = 0.1$. In contrast to the near constancy of this ratio as shown in the figure, $T_c/\Lambda_{\overline{MS}}$ computed from the same data using the bare lattice definition of g falls from 3.9 to 2.8 in going from $N_t = 4$ to 8. To extend this test to other values of m/T_c we have to interpolate between plaquette values for various quark masses. An upper limit for the systematic error in this procedure is the actual change in $\mathcal{P}$ between extreme values of the quark masses at which they are measured. This varies between a few parts in a thousand at $\beta = 5.26$ to about 2% at $\beta = 6$. This uncertainty in $\mathcal{P}$ translates into a similar magnitude of uncertainty in $T_c/\Lambda_{\overline{MS}}$ and is much smaller than the change when using $6/\beta$ across a similar range of quark masses. Excellent scaling of $T_c/\Lambda_{\overline{MS}}$ is seen also for other values of m/T_c.

In order to obtain a physically relevant value of T_c, it is necessary to extrapolate $T_c/\Lambda_{\overline{MS}}$ to measured values of the hadron masses. It would be most interesting to perform this extrapolation in the quark or pion masses. However, this needs control over the critical exponents of the theory— a task we do not attempt here. Instead we choose to extrapolate T_c to the physical region in terms of M_ρ [21]. There are two reasons for this. First, the ratio T_c/M_ρ is known to be nearly constant. Secondly, M_ρ is quite sensitive to finite lattice spacing effects. The linearity of the plot of $T_c/\Lambda_{\overline{MS}}$ against $M_\rho/\Lambda_{\overline{MS}}$ in Figure 4 then indicates that, for the chosen data set, finite lattice spacing effects in M_ρ are under reasonable control.

The value of $\Lambda_{\overline{MS}}$ also needs to be specified. This depends on how many active flavours are present at the scale under consideration. The world average of $\Lambda^{(5)}_{\overline{MS}}$ (at scales high enough for 5 active flavours) is 219^{+25}_{-23} [4]. At lower scales, with only three active flavours,

$\Lambda^{(3)}_{\overline{MS}} = 343^{+31}_{-28}$, using the prescription of [4] to match across flavour thresholds.

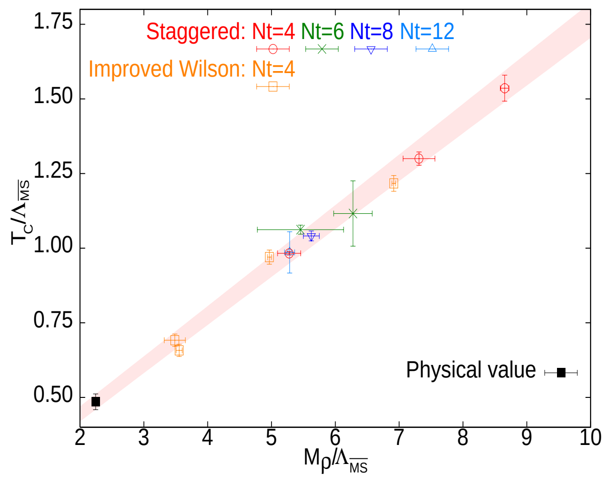

FIG. 4. $T_c/\Lambda_{\overline{MS}}$ plotted against $M_\rho/\Lambda_{\overline{MS}}$ and extrapolated to the physical value. The band shows the 1-σ errors on the best fit line to the measurements.

The physical value of T_c is obtained by extrapolating to the real-world value of $M_\rho/\Lambda^{(3)}_{\overline{MS}}$. This is done by fitting a straight line to the data including staggered and improved Wilson quarks [22]. We have taken the two sets of data together since they both have cutoff effects of order a^2. The extrapolation gives $T_c/\Lambda_{\overline{MS}} = 0.487 \pm 0.023$, and therefore

$$T_c = 167 \pm 9^{+15}_{-14}. \qquad (9)$$

The first error is purely from extrapolation and the second set from the errors on $\Lambda^{(3)}_{\overline{MS}}$. As before, it is possible to add them in quadrature. This result is consistent with a recent estimate using $O(4)$ critical indices to scale T_c as the pion mass is taken to its physical value [23]. In these determinations one has to keep systematic errors in mind. Currently the largest source of such uncertainty is whether extrapolating one hadron mass to its measured value also takes all other hadron masses to their correct values. Until this issue is settled, all estimates of T_c must be considered preliminary.

In summary, we demonstrated that the lattice data on strong interaction thermodynamics obey QCD scaling relations very well and allow continuum physics to be extracted on fairly coarse lattices. As a result, $T_c/\Lambda_{\overline{MS}}$ can be measured with precision of better than four parts in a thousand in the pure gauge theory. The main uncertainty in T_c thus comes from errors in $\Lambda_{\overline{MS}}$ (see eq. 6). Simulations including dynamical quarks, when extrapolated to physical values of the ρ meson mass give statistical errors of about 4% in $T_c/\Lambda_{\overline{MS}}$. The pressure, energy density and the speed of sound can also be measured precisely for $T > 1.3T_c$. A study of finite volume effects on thermodynamics closer to T_c will be reported elsewhere.

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